MTH6132, Relativity
Solutions for Problem Set 2
Due 17$^{\text{th}}$ October 2018

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Parametrized Curves

We are given a point $p \mapsto (x, y, z) = (1, 0, -1)$, and a curve $(x(\lambda), y(\lambda), z(\lambda)) = (\lambda, (1-\lambda)^2, -\lambda)$.

a) [1 Point]
The tangent vector $\vec{u}(\lambda) \mapsto (\frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda})$ at all points along the curve is $\vec{u}(\lambda) \mapsto (1, -2(1-\lambda), -1)$.

b) [2 Points]
The point p is located at $\lambda = 1$. The tangent vector at point p is thus $\vec{u}(1) \mapsto (1, 0, -1)$.

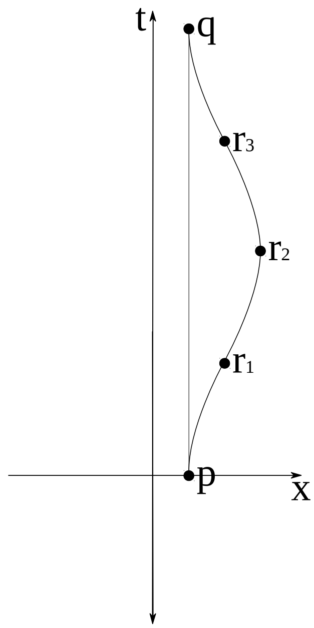
c) [2 Points]
Defining a scalar field $f(x, y, z) = x^2 + y^2 - yz$, its values along the curve are

$$\begin{aligned}
\frac{df}{d\lambda} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \lambda} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \lambda} \\
&= u^x \frac{\partial f}{\partial x} + u^y \frac{\partial f}{\partial y} + u^z \frac{\partial f}{\partial z} \\
&= (1)(2x) + (-2(1-\lambda))(2y - z) + (-1)(-y). \\
&= 2\lambda - 2(1-\lambda)(2(1-\lambda)^2 + \lambda) + (1-\lambda)^2. \\
&= 4\lambda^3 - 9\lambda^2 + 10\lambda - 3.
\end{aligned}$$

This is the simplest example of a *Lie derivative* $\mathcal{L}_{\vec{u}}$ in the direction of $\vec{u}(\lambda)$, which is the tangent vector to some curve parametrized by λ. What you computed in (c) was the Lie derivative $\mathcal{L}_{\vec{u}} f$ of a scalar field f in the direction of $\vec{u}(\lambda)$.

The Twin Paradox Revisited

We are given a frame $F : (t, x, y, z)$ that Sol and Alpha Centauri are at rest in, and a frame $F' : (\tau, X, Y, Z)$ of a ship that is traveling with constant four-acceleration in the $+x$ direction. The ship starts its journey at point p labeled by $t(0) = 0, x(0) = 1/\alpha$ in the F frame, with initial four-velocity $\vec{u}(0) \mapsto (u^t(0), u^x(0), 0, 0) = (1, 0, 0, 0)$. During its journey from point p to some point r_1, its four-acceleration is a constant $\vec{a} \mapsto (a^\tau, a^X, a^Y, a^Z) = (0, \alpha, 0, 0)$ as written in the F' frame, for $\alpha > 0$.



a) [10 Points]
Using $\langle \vec{u}(\tau), \vec{a} \rangle = 0$, $\langle \vec{u}(\tau), \vec{u}(\tau) \rangle = -1$, and $\langle \vec{a}, \vec{a} \rangle = \alpha^2$, we are asked to show that $a^t = \alpha u^x$ and $a^x = \alpha u^t$.

A healthy reaction is to, at some point r along the ship's worldline, Lorentz transform the four-velocity components to compare how they are written in the F frame and the F' frame. Notice that such a Lorentz transformation has a boost parameter β that varies $\beta = \beta(\tau)$ as you vary the point r along the ship's worldline, labelled by $t(\tau), x(\tau)$ in the F frame.

Let us perform this Lorentz transformation on the components of the four-velocity to see how far it gets us

$$\begin{aligned} u^t(\tau) &= \gamma \cancelto{1}{u^\tau(\tau)} + \gamma\beta \cancelto{0}{u^X(\tau)} \\ u^x(\tau) &= \gamma\beta \cancelto{1}{u^\tau(\tau)} + \gamma \cancelto{0}{u^X(\tau)}, \end{aligned}$$

so we conclude that $u^t(\tau) = \gamma(\tau)$ and $u^x(\tau) = \gamma(\tau)\beta(\tau)$, where $\gamma \equiv \sqrt{1-\beta^2}$. We have no access to the function $\beta(\tau)$ so we can go no further along this line of inquiry. Nevertheless, since $\gamma = \gamma(\tau) > 0$ and $\beta = \beta(\tau) > 0$ for all τ along the ship's worldline from p to r_1, we can still conclude that

$$u^t(\tau) > 0, u^x(\tau) > 0. \tag{1}$$

We can perform the same Lorentz transformation on the components of the four-acceleration

$$\begin{aligned} a^t(\tau) &= \gamma \cancelto{0}{a^\tau} + \gamma\beta \cancelto{\alpha}{a^X} \\ a^x(\tau) &= \gamma\beta \cancelto{0}{a^\tau} + \gamma \cancelto{\alpha}{a^X}, \end{aligned}$$

to conclude that

$$a^t > 0, a^x > 0. \tag{2}$$

With that out of the way, let us write out the relations between the vectors that define the ship's worldline, because doing so will show us how to solve for $u^t(\tau), u^x(\tau)$, and thus effectively find the unknown $\beta(\tau)$.

Orthogonality of four-velocity and four-acceleration in all frames $\langle \vec{u}(\tau), \vec{a} \rangle = 0$: this gives us

$$-u^t a^t + u^x a^x = 0. \tag{3}$$

Norm of four-velocity in all frames $\langle \vec{u}(\tau), \vec{u}(\tau) \rangle = -1$: this gives us

$$-(u^t)^2 + (u^x)^2 = -1. \tag{4}$$

Norm of four-acceleration in all frames $\langle \vec{a}, \vec{a} \rangle = \alpha^2$: this gives us

$$-(a^t)^2 + (a^x)^2 = \alpha^2. \tag{5}$$

Starting with (5), use (3) to replace a^x with u^t, u^x, a^t, then use (4) to find

$$\alpha^2 = -(a^t)^2 + \left(\frac{u^t a^t}{u^x}\right)^2 = (a^t)^2 \left(-1 + \frac{(u^t)^2}{(u^x)^2}\right) = (a^t)^2 \left(\cancelto{1}{\frac{-(u^x)^2 + (u^t)^2}{(u^x)^2}}\right),$$

so remembering (1) and (2) that lead us to pick the "+″ sign when taking the square root of the squares, we conclude that $a^t = \alpha u^x$.

Similarly, starting with (5), use (3) to replace a^t with u^t, u^x, a^x, then use (4) to find

$$\alpha^2 = -\left(\frac{u^x a^x}{u^t}\right)^2 + (a^x)^2 = (a^x)^2 \left(-\frac{(u^x)^2}{(u^t)^2} + 1\right) = (a^x)^2 \left(\cancelto{1}{\frac{-(u^x)^2 + (u^t)^2}{(u^t)^2}}\right),$$

so remembering (1) and (2) that lead us to pick the "+″ sign when taking the square root of the squares, we conclude that $a^x = \alpha u^t$.

b) [10 Points]

We are to write the result of (a) as the linear system

$$\begin{aligned} \frac{du^t}{d\tau} &= \alpha u^x \\ \frac{du^x}{d\tau} &= \alpha u^t, \end{aligned} \tag{6}$$

with initial conditions $u^t(0) = 1, u^x(0) = 0$, and solve it to find $u^t(\tau) = \cosh(\alpha\tau)$, $u^x(\tau) = \sinh(\alpha\tau)$.

Writing down this linear system of ordinary differential equations relies simply on remembering that in any inertial frame $F : (t, x, y, z)$, the components of the four-acceleration are $\vec{a} \mapsto (a^t, a^x, a^y, a^z) = (\frac{du^t}{d\tau}, \frac{du^x}{d\tau}, \frac{du^y}{d\tau}, \frac{du^z}{d\tau})$.

Solving this linear system is a standard problem in ordinary differential equations. Since this are already familiar to you, here I will reintroduce the concepts required to solve this problem from scratch, but using index notation. I do this in order to show you how familiar equations look like in index notation, as seeing this new notation used explicitly in a familiar setting may be of some benefit to you.

In $x^\mu = (t, x)$ coordinates[1], the linear system (6) can be written succinctly as

$$\frac{du^\mu}{d\tau} = M^\mu{}_\nu u^\nu, \tag{7}$$

where the components $M^\mu{}_\nu$ of $\mathbf{M}$ in these coordinates are $M^t{}_t = 0$, $M^t{}_x = \alpha$, $M^x{}_t = \alpha$, $M^x{}_x = 0$.

[1]Note that in the following, any x^α with Greek indices $\alpha, \beta, ..., \mu, ...$ will refer to these coordinates $x^\alpha = (t, x)$

The components $M^\mu{}_\nu$ form a matrix that is real and symmetric, and thus it is diagonalizable i.e. there exists some other coordinates $x^a = (T, X)$ such that[2]

$$\frac{du^a}{d\tau} = M^a{}_b u^b \tag{8}$$

where the components $M^a{}_b$ of $\mathbf{M}$ in these new coordinates are "on the diagonal" $M^T{}_T = \lambda_1$, $M^T{}_X = 0$, $M^X{}_T = 0$, $M^X{}_X = \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

In other words $\exists \vec{u}_1, \vec{u}_2$ (which we henceforth denote by $\vec{u}_i$ for $i = 1, 2$ with components $(u_i)^T, (u_i)^X$ which we henceforth denote by $(u_i)^a$) with components $(u_1)^T = 1$, $(u_1)^X = 0$ and $(u_2)^T = 0$, $(u_2)^X = 1$, for which

$$M^a{}_b (u_i)^b = \lambda_i (u_i)^a \tag{9}$$

In (9), the λ_i are known as *eigenvalues* and the $\vec{u}_i$ for $i = 1, 2$ are known as *eigenvectors*. We have written (9) in $x^a = (T, X)$ coordinates, but the equation takes on exactly the same form in $x^\mu = (t, x)$ coordinates[3] i.e.

$$M^\mu{}_\nu (u_i)^\nu = \lambda_i (u_i)^\nu. \tag{15}$$

though in these coordinates, the components $(u_i)^t, (u_i)^x$ are as yet unknown.

We can find the eigenvalues λ_1, λ_2 by first rearranging (15)

$$(M^\mu{}_\nu - \lambda_i \delta^\mu_\nu)(u_i)^\nu = 0, \tag{16}$$

which holds for non-zero vectors $\vec{u}_i$, so it must be that

$$\det(M^\mu{}_\nu - \lambda_i \delta^\mu_\nu) = 0. \tag{17}$$

(17) is known as the *characteristic equation* of $\mathbf{M}$, For the values $M^t{}_t = 0$, $M^t{}_x = \alpha$, $M^x{}_t = \alpha$, $M^x{}_x = 0$, this equation reads

$$(\lambda_i)^2 - \alpha^2 = 0, \tag{18}$$

so we conclude that the eigenvalues of this system are $\lambda_1 = \alpha$ and $\lambda_2 = -\alpha$.

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[2] Note that in the following, any x^a with Latin indices $a, b, ..., m, ...$ will refer to these coordinates $x^a = (T, X)$.

[3] To prove (15), it suffices to know how all the relevant objects transform under a change of coordinates $x^\mu \to x^a$:

$$\begin{aligned} M^a{}_b &= \frac{\partial x^a}{\partial x^\mu} M^\mu{}_\nu \frac{\partial x^\nu}{\partial x^b} \\ u^a &= \frac{\partial x^a}{\partial x^\mu} u^\mu, \end{aligned} \tag{10}$$

and the identity

$$\frac{\partial x^a}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^b} = \frac{\partial x^a}{\partial x^b} = \delta^a_b, \tag{11}$$

where the *Kronecker delta symbol* has values $\delta^a_b = 1$ when $a = b$ and $\delta^a_b = 0$ when $a \neq b$. The left hand side of (9) is

$$\begin{aligned} M^a{}_b (u_i)^b &= \left(\frac{\partial x^a}{\partial x^\mu} M^\mu{}_\nu \frac{\partial x^\nu}{\partial x^b} \right) \left(\frac{\partial x^b}{\partial x^\beta} (u_i)^\beta \right) \\ &= \frac{\partial x^a}{\partial x^\mu} M^\mu{}_\nu \delta^\nu_\beta (u_i)^\beta \\ &= \frac{\partial x^a}{\partial x^\mu} M^\mu{}_\nu (u_i)^\nu, \end{aligned} \tag{12}$$

and the right hand side of (9) is

$$\lambda_i (u_i)^a = \lambda_i \left(\frac{\partial x^a}{\partial x^\alpha} (u_i)^\alpha \right), \tag{13}$$

so contracting both (12) and (13) by $\frac{\partial x^\mu}{\partial x^a}$, and equating them with each other gives

$$M^\mu{}_\nu (u_i)^\nu = \lambda_i \delta^\mu_\alpha (u_i)^\alpha, \tag{14}$$

which completes the proof of (15).

To find the eigenvector $\vec{u}_1$ that corresponds to λ_1, evaluate (16) with $\lambda_1 = \alpha$ and $M^t{}_t = 0$, $M^t{}_x = \alpha$, $M^x{}_t = \alpha$, $M^x{}_x = 0$ to find that $(u_1)^t = (u_1)^x$, and remembering that $\langle \vec{u}_1, \vec{u}_1 \rangle = -1$ we conclude that $\vec{u}_1 \mapsto ((u_1)^t, (u_1)^x) = \frac{1}{\sqrt{2}}(1, 1)$.

Similarly, to find the eigenvector $\vec{u}_2$ that corresponds to λ_2, evaluate (16) with $\lambda_2 = -\alpha$ and $M^t{}_t = 0$, $M^t{}_x = \alpha$, $M^x{}_t = \alpha$, $M^x{}_x = 0$ to find that $(u_2)^t = -(u_2)^x$, and remembering that $\langle \vec{u}_2, \vec{u}_2 \rangle = -1$ we conclude that $\vec{u}_2 \mapsto ((u_2)^t, (u_2)^x) = \frac{1}{\sqrt{2}}(-1, 1)$.

We now know that $(u_1)^t = \frac{1}{\sqrt{2}}$, $(u_1)^x = \frac{1}{\sqrt{2}}$ and $(u_2)^t = -\frac{1}{\sqrt{2}}$, $(u_2)^x = \frac{1}{\sqrt{2}}$ in $x^\mu = (t, x)$ coordinates and $(u_1)^T = 1$, $(u_1)^X = 0$ and $(u_2)^T = 0$, $(u_2)^X = 1$ in $x^a = (T, X)$ coordinates. This gives us enough information to reconstruct the four components $\frac{\partial t}{\partial T}$, $\frac{\partial x}{\partial T}$, $\frac{\partial t}{\partial X}$, $\frac{\partial x}{\partial X}$ of the Jacobian $\frac{\partial x^\mu}{\partial x^a}$ of the transformation $x^a \to x^\mu$ because we know that vectors transform by

$$u^\mu = \frac{\partial x^\mu}{\partial x^a} u^a, \tag{19}$$

so we find that[4] $\frac{\partial t}{\partial T} = \frac{1}{\sqrt{2}}$, $\frac{\partial x}{\partial T} = \frac{1}{\sqrt{2}}$, $\frac{\partial t}{\partial X} = -\frac{1}{\sqrt{2}}$, $\frac{\partial x}{\partial X} = \frac{1}{\sqrt{2}}$.

We will use this to first solve (8) to find u^T, u^X in $x^a = (T, X)$ coordinates, then transform via (19) to find u^t, u^x in $x^\mu = (t, x)$ coordinates.

Solving (8) for u^T, u^X in $x^a = (T, X)$ is simply a matter of integrating. It's T component can be rearranged to

$$\alpha = \frac{1}{u^T} \frac{du^T}{d\tau}, \tag{20}$$

so that integration by substitution gives

$$\alpha\tau + const. = \int d\tau \left(\frac{1}{u^T} \frac{du^T}{d\tau} \right) = \int du^T \frac{1}{u^T} = \ln u^T, \tag{21}$$

so we conclude that $u^T = A \exp(\alpha\tau)$ for some constant A, and using the same steps with $\alpha \to -\alpha$ and $u^T \to u^X$ in (20) we conclude that $u^X = B \exp(-\alpha\tau)$ for some constant B.

Finding u^t, u^x is now simply a question of using (19) to transform from u^T, u^X to u^t, u^x

$$\begin{aligned} u^t &= \frac{\partial t}{\partial T} u^T + \frac{\partial t}{\partial X} u^X = \frac{A}{\sqrt{2}} \exp(\alpha\tau) - \frac{B}{\sqrt{2}} \exp(-\alpha\tau) \\ u^x &= \frac{\partial x}{\partial T} u^T + \frac{\partial x}{\partial X} u^X = \frac{A}{\sqrt{2}} \exp(\alpha\tau) + \frac{B}{\sqrt{2}} \exp(-\alpha\tau), \end{aligned}$$

and applying the initial conditions $u^t(0) = 1$, $u^x(0) = 0$ shows us that $A = \frac{1}{\sqrt{2}}$ and $B = -\frac{1}{\sqrt{2}}$. We thus conclude that

$$\begin{aligned} u^t &= \frac{1}{2} \left(\exp(\alpha\tau) + \exp(-\alpha\tau) \right) = \cosh(\alpha\tau) \\ u^x &= \frac{1}{2} \left(\exp(\alpha\tau) - \exp(-\alpha\tau) \right) = \sinh(\alpha\tau). \end{aligned}$$

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[4]To find this result, write out (19) explicitly component by component to find that

$$\begin{aligned} \cancelto{\frac{1}{\sqrt{2}}}{(u_1)^t} &= \frac{\partial t}{\partial x^a} u^a = \frac{\partial t}{\partial T} \cancelto{1}{(u_1)^T} + \frac{\partial t}{\partial X} \cancelto{0}{(u_1)^X} \\ \cancelto{\frac{1}{\sqrt{2}}}{(u_1)^x} &= \frac{\partial x}{\partial x^a} u^a = \frac{\partial x}{\partial T} \cancelto{1}{(u_1)^T} + \frac{\partial x}{\partial X} \cancelto{0}{(u_1)^X} \\ \cancelto{-\frac{1}{\sqrt{2}}}{(u_2)^t} &= \frac{\partial t}{\partial x^a} u^a = \frac{\partial t}{\partial T} \cancelto{0}{(u_2)^T} + \frac{\partial t}{\partial X} \cancelto{1}{(u_2)^X} \\ \cancelto{\frac{1}{\sqrt{2}}}{(u_2)^x} &= \frac{\partial x}{\partial x^a} u^a = \frac{\partial x}{\partial T} \cancelto{0}{(u_2)^T} + \frac{\partial x}{\partial X} \cancelto{1}{(u_2)^X}. \end{aligned}$$

c) [2 Points]
We are to use the result of (b) with initial conditions $t(0) = 0, x(0) = 1/\alpha$ to show that the worldline of the ship is $t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)$.

Writing down this worldline relies simply on remembering that in any frame $F : (t, x, y, z)$, the components of the four-velocity are $\vec{u} \mapsto (u^t, u^x, u^y, u^z) = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau})$.

d) [1 Point]
We are to find the elapsed proper time $\Delta\tau$ between p and q experienced by the twin in the ship, as in the spacetime diagram above, given that the point r_1 has label $(t(\tau_{halfway}), x(\tau_{halfway}))$ in frame F with $x(\tau_{halfway}) = \frac{1}{\alpha}\cos\alpha$.

The result in part (c) for $x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)$ implies that $\tau_{halfway} = 1$. Since the worldline from point p to point r_1 is one-fourth of the entire journey, with each part of the trip is characterized by the same constant acceleration/decceleration of α i.e. part 1 from point p to point r_1, part 2 from point r_1 to point r_2, part 3 from point r_2 to point r_3, part 4 from point r_3 to point q, each of these four parts have the same proper time elapsed.

We then conclude that $\Delta\tau$ from point p to point q along the worldline whose tangent vector is $\vec{u} \mapsto (u^t(\tau), u^x(\tau), 0, 0)$ with $u^t(\tau) = \cosh(\alpha\tau)$, $u^x(\tau) = \sinh(\alpha\tau)$ is $\Delta\tau = 4$.

e) [1 Point]
We are to find the elapsed proper time Δt between p and q experienced by the twin at rest in frame F.

The result in part (c) for $t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$ and the result in part (d) for $\tau_{halfway} = 1$ imply that point r_1 is labeled by $(t(\tau_{halfway}), x(\tau_{halfway}))$ in frame F with $t(\tau_{halfway}) = \frac{1}{\alpha}\sinh(\alpha)$.

We then conclude that Δt from point p to point q along the worldline whose tangent vector is $\vec{u} \mapsto (u^t, u^x, 0, 0) = (1, 0, 0, 0)$ is $\Delta t = \frac{4}{\alpha}\sinh(\alpha)$.

f) [1 Point]
We are to make a conjecture about which worldline maximizes the elapsed proper time experienced by an observer traveling from point p to point q.

We know that between any two fixed endpoints p and q, the worldline that maximizes the elapsed proper time is the "straight line".