MTH6132, Relativity

Problem Set 9

Due Anytime

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The Greatest Hits of the 2000s

(NB: the text below is taken verbatim from Exam-2009.pdf, Problem 9.b. with notation updates.)
You are given that the trajectory of light rays in an equatorial plane in the Schwarzschild metric, to zeroth order approximation, is given by

$$u = L \cos \varphi$$

where $u = \frac{1}{r}$ and $\frac{1}{L}$ is the smallest value of r attained, corresponding to $\varphi = 0$. If a light ray is emitted at $r = R$ and travels via $r = \frac{1}{L}$ to $r = R$ again, show that he coordinate time t between emission and reception is, to this approximation, given by

$$T = 2 \int_{\frac{1}{L}}^{R} \left(\frac{1}{f(r)^2} + \frac{1}{f(r)(L^2 r^2 - 1)} \right)^{\frac{1}{2}} dr, \qquad \text{where } f(r) = 1 - 2GM/r.$$

The Greatest Hits of the 2010s

(NB: the text below is taken verbatim from Exam-2016.pdf, Problem 1.)
Let F and F' denote two inertial reference systems moving with velocity v with respect to each other. In F, two events occur simultaneously at $t = 0$, separated by a distance X along the x-axis. The time interval between the events in F' is T.

(a) Draw a 2-dimensional spacetime diagram describing the situation, including both F and F'. You may assume units for which $c = 1$.

(b) Show that the spatial distance between the two events in F' is $\sqrt{X^2 + T^2}$.

(c) Determine the relative velocity v of the frames F, F' in terms of X and T. You may assume $c = 1$ in your calculations.

Gravitational Waves and Circular Rings

A circular ring is lying on the x, y plane at $z = 0$ and centered at $x = 0, y = 0$. Your attention is drawn to four spacelike curves connecting diametrically opposite points on this circular ring: the "vertical line" along the y axis, the "horizontal line" along the x axis, the "upward sloping diagonal" along $y = x$, and the "downward sloping diagonal" along $y = -x$. A gravitational wave passes through this circular ring written in coordinates $x^a = (t, x, y, z)$

$$g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos(\omega t - kz),$$

(a) Calculate the arc lengths $\sigma_{vertical}$, $\sigma_{horizontal}$, σ_{upward}, $\sigma_{downward}$ of all four spacelike curves described above, in terms of $h_+, h_\times$ and as a function of t.

(b) Thus describe the effect of the gravitational wave on the circular ring.