MTH6132, Relativity

Problem Set 6

Due 21st November 2018

Rodrigo Panosso Macedo

Levi-Civita connection and Geodesics

1. Consider the 2-sphere of radius a having line element

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2.$$

(a) Calculate the connection coefficients to show that the non-vanishing values are:

$$\Gamma^1{}_{22} = -\cos\theta \sin\theta, \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta.$$

(b) Write down the geodesic equations on the unit sphere $\mathbb{S}^2$. Show that one solution to the system of equations is $\theta = \lambda$, $\varphi =$ constant. Provide a geometric interpretation of this solution curve, justifying your answer.

2. Find the geodesic equations on a cylinder of radius a. The metric in coordinates (z, φ) is $ds^2 = dz^2 + a^2 d\varphi^2$. Why is the result so simple?

3. V^a is called a *Killing vector* if it satisfies the following equation

$$2\nabla_{(a}V_{b)} = \nabla_a V_b + \nabla_b V_a = 0.$$

Let X^a be the tangent vector to a geodesic, let V^a be a Killing vector, and define the scalar E by

$$E \equiv V_a X^a = g_{ab} V^a X^b.$$

Show that E is conserved along geodesics, i.e. that $X^a \nabla_a E = 0$.

4. The metric for a particular 2-dimensional spacetime is given by

$$ds^2 = -e^{2Ar} dt^2 + dr^2$$

where A is an arbitrary constant. Use the Euler-Lagrange equations to calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric.