MTH6132, Relativity

Solutions to Problem Set 6

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1. [8 points]

(a) See lecture notes

(b) Geodesic equations $\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$. Explicitly it reads with $x^a = (\theta, \varphi)$

- $a = 1$

$$\begin{aligned} \ddot{x}^1 + \Gamma^1{}_{11}(\dot{x}^1)^2 + 2\Gamma^1{}_{12}\dot{x}^1\dot{x}^2 + \Gamma^1{}_{22}(\dot{x}^2)^2 &= 0 \\ \ddot{\theta} + \Gamma^1{}_{11}\dot{\theta}^2 + 2\Gamma^1{}_{12}\dot{\theta}\dot{\varphi} + \Gamma^1{}_{22}\dot{\varphi} &= 0 \qquad (1) \end{aligned}$$

- $a = 2$

$$\begin{aligned} \ddot{x}^2 + \Gamma^2{}_{11}(\dot{x}^1)^2 + 2\Gamma^2{}_{12}\dot{x}^1\dot{x}^2 + \Gamma^2{}_{22}(\dot{x}^2)^2 &= 0 \\ \ddot{\varphi} + \Gamma^2{}_{11}\dot{\theta}^2 + 2\Gamma^2{}_{12}\dot{\theta}\dot{\varphi} + \Gamma^2{}_{22}\dot{\varphi} &= 0 \qquad (2) \end{aligned}$$

Thus the geodesic equations on the unit sphere are given by

$$\begin{aligned} \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 &= 0 \\ \ddot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} &= 0. \end{aligned}$$

Equations are trivially satisfied by $\varphi =$constant and $\theta = \lambda \in [0, 2\pi)$. Curves passing through the north ($\theta = 0$) and south pole ($\theta = \pi$), and dividing the sphere into two equal hemispheres. Analogue to the Longitude curves on the globe.

2. [6 points] The metric $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}$ is constant. Since, $\partial_c g_{ab}$ always vanishes, we have $\Gamma^a{}_{bc} = 0$. Geodesic equations is just

$$\ddot{z} = 0, \quad \ddot{\varphi} = 0. \qquad (3)$$

The simplicity arises because the space is flat. A cylinder of radius a is just the flat Euclidian space with coordinates (x, z) with one compact direction, i.e., the surface $x = 2\pi\, a$ is identified with the origin $x = 0$.

3. [6 points] X^a is the tangent vector to a geodesic, thus $X^a\nabla_a X^b = 0$.

Killing equation: $\nabla_{(a}V_{b)} = \nabla_a V_b + \nabla_b V_a = 0$, thus $\nabla_a V_b = -\nabla_b V_a$, i.e., $\nabla_a V_b$ is anti-symmetric.

Given $E = V_a X^a$, we get

$$\begin{aligned} X^a\nabla_a E &= X^a\nabla_a\left(V_b X^b\right) \\ &= \underbrace{X^a X^b}_{\text{Sym.}}\ \underbrace{\left(\nabla_a V_b\right)}_{\text{Anti-Sym}} \overset{\nearrow 0}{} + \underbrace{X^a\left(\nabla_a X^b\right)}_{\text{Geod. Eq.}} \overset{\nearrow 0}{} V_b \\ &= 0. \end{aligned}$$

4. [10 points] Identifying the coordinates $x^0 = t$, $x^1 = r$. The Lagragian is

$$L = -e^{2Ar}\dot{t}^2 + \dot{r}^2.$$

Now, for the t components one has that

$$\frac{\partial L}{\partial t} = 0, \quad \frac{\partial L}{\partial \dot{t}} = -2e^{2Ar}\dot{t}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-2e^{2Ar}\dot{t}\right) = \ddot{t}e^{2Ar} + 2Ae^{2Ar}\dot{r}\dot{t} = 0.$$

Thus, the Euler-Lagrange equation is given by

$$\ddot{t} + 2A\dot{r}\dot{t} = 0.$$

Now, comparing with the geodesic equation

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \quad \overset{x^0=t}{\Longrightarrow} \quad \ddot{t} + \Gamma^0{}_{00}\dot{t}^2 + 2\Gamma^0{}_{01}\dot{t}\dot{r} + \Gamma^0{}_{11}\dot{r}^2 = 0.$$

Comparing one gets

$$\Gamma^0_{00} = 0, \quad \Gamma^0{}_{01} = \Gamma^0{}_{10} = A, \quad \Gamma^0{}_{11} = 0.$$

For the r components one has that

$$\frac{\partial L}{\partial r} = -2Ae^{2Ar}\dot{t}^2, \quad \frac{\partial L}{\partial \dot{r}} = 2\dot{r}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) = 2\ddot{r}.$$

Hence, the Euler-Lagrange equation is given by

$$\ddot{r} + Ae^{2Ar}\dot{t}^2 = 0.$$

Again, comparing with the geodesic equation

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \quad \overset{x^1=r}{\Longrightarrow} \quad \ddot{r} + \Gamma^1{}_{00}\dot{t}^2 + 2\Gamma^1{}_{01}\dot{t}\dot{r} + \Gamma^1{}_{11}\dot{r}^2 = 0.$$

gives,

$$\Gamma^1{}_{11} = 0, \quad \Gamma^1{}_{01} = \Gamma^1{}_{10} = 0. \quad \Gamma^1{}_{00} = Ae^{2Ar}.$$