MTH6132, Relativity

Solutions to Problem Set 5

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1. [8 points]

$$
\begin{aligned}
&\nabla_b \Phi = \partial_b \Phi, \quad \text{Define the scalar according to scalar product } \Phi = V_a W^a \\
&\nabla_b (V_a W^a) = \partial_b (V_a W^a) \quad \text{Use Leibniz rule} \\
&(\nabla_b V_a) W^a + V_a \underbrace{(\nabla_b W^a)}_{\partial_b W^a + \Gamma^a{}_{cb} W^c} = (\partial_b V_a) + V_a (\partial_b W^a) \quad \text{Use covariant derivative of contravariant vector} \\
&(\nabla_b V_a) W^a + \cancel{V_a \partial_b W^a} + \Gamma^{\cancelto{c}{a}}{}_{\cancelto{a}{c}b} V_{\cancelto{c}{a}} W^{\cancelto{a}{c}} = (\partial_b V_a) + \cancel{V_a (\partial_b W^a)} \quad \text{Re-label dummy indices } a \to c,\ c \to a \\
&\left(\nabla_b V_a - \partial_b V_a + \Gamma^c{}_{ab} V_c \right) W^a = 0 \quad \text{Factor out generic vector } W^a \\
&\nabla_b V_a = \partial_b V_a - \Gamma^c{}_{ab} V_c .
\end{aligned}
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2. [10 points] Given $\nabla^a T_{ab} = 0$ and T_{ab} symmetric and

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\begin{aligned}
&\nabla_{(a} X_{b)} = 0 \quad \text{Definition of symmetry operation -- see problem set 3} \\
&(\nabla_a X_b + \nabla_b X_a)/2 = 0 \\
&\nabla_a X_b = -\nabla_b X_a .
\end{aligned}
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Thus $\nabla_a X_b$ is anti=symmetric. Let $V_a = T_{ab} X^b$, then

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\begin{aligned}
\nabla^a V_a &= \nabla^a (T_{ab} X^b) \quad \text{Use Leibniz rule} \\
&= \cancelto{0}{(\nabla^a T_{ab})} X^b + \cancelto{0}{\underbrace{T_{ab}}_{\text{Symmetric}} \underbrace{(\nabla^a X^b)}_{\text{Anti-symmetric}}}
\end{aligned}
$$

Symmetric/Anti-symmetric contraction — see problem set 3

3.[4 points]

(a) $\nabla_a S_b{}^c = \partial_a S_b{}^c + \Gamma^c{}_{da} S^d{}_b - \Gamma^d{}_{ba} S^c{}_d$

(b) Use expression above for $S_b{}^c = \delta_b{}^c$

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\begin{aligned}
&\nabla_a \delta_b{}^c = \cancelto{0}{\partial_a \delta_b{}^c} + \Gamma^c{}_{da} \delta^d{}_b - \Gamma^d{}_{ba} \delta^c{}_d, \quad \delta_b{}^c \text{ is either 1 or 0. Derivative vanishes; Use Delta definition} \\
&\nabla_a \delta_b{}^c = \Gamma^c{}_{ba} - \Gamma^c{}_{ba} = 0
\end{aligned}
$$

(b) Dimensions 4, $a = 1 \cdots 4 \Rightarrow \delta_a{}^a = \cancelto{1}{\delta_1{}^1} + \cancelto{1}{\delta_2{}^2} + \cancelto{1}{\delta_3{}^3} + \cancelto{1}{\delta_4{}^4} = 4.$

Notice that $\delta_a{}^a$ correspond to the trace of the identity matrix. Therefore $\delta_a{}^a = d$, with d the dimension of the matrix.

4. [8 points] See lecture notes: Geodesic equation