MTH6132, Relativity

Solutions to Problem Set 4

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Metric: coordinate change and matrix representation

Change of the differential form

$$dx^a = \frac{\partial x^a}{\partial x'^a} dx'^a = \frac{\partial x^a}{\partial x'^0} dx'^0 + \frac{\partial x^a}{\partial x'^1} dx'^1 + \frac{\partial x^a}{\partial x'^2} dx'^2 + \frac{\partial x^a}{\partial x'^3} dx'^3. \tag{1}$$

1. [7 points] Cartesian coordinates $x^a = (t, x, y, z)$; spherical coordinates $x'^a = (t', r, \theta, \varphi)$ with

$$t = t', \quad x = r \sin\theta \sin\varphi, \quad y = r \sin\theta \cos\varphi, \quad z = r\cos\theta$$

- Use eq. (1) for $a = 0$:

$$\begin{aligned} dt &= \frac{\partial t}{\partial t'} dt' + \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial \theta} d\theta + \frac{\partial t}{\partial \varphi} d\varphi \\ &= dt' \\ dt^2 &= dt'^2 \end{aligned}$$

- Use eq. (1) for $a = 1$:

$$\begin{aligned} dx &= \frac{\partial x}{\partial t'} dt' + \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi \\ &= \sin\theta \sin\varphi \, dr + r\cos\theta \sin\varphi \, d\theta + r \sin\theta \cos\varphi \, d\varphi \\ dx^2 &= \sin^2\theta \sin^2\varphi \, dr^2 + r^2 \cos^2\theta \sin^2\varphi \, d\theta^2 + r^2 \sin^2\theta \cos^2\varphi \, d\varphi^2 \\ &\quad + 2r \sin\theta \cos\theta \sin^2\varphi \, dr d\theta + 2r \sin^2\theta \sin\varphi \cos\varphi \, dr d\varphi + 2r^2 \cos\theta \sin\theta \sin\varphi \cos\varphi \, d\theta d\varphi \end{aligned}$$

- Use eq. (1) for $a = 2$:

$$\begin{aligned} dy &= \frac{\partial y}{\partial t'} dt' + \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi \\ &= \sin\theta \cos\varphi \, dr + r \cos\theta \cos\varphi \, d\theta - r \sin\theta \sin\varphi \, d\varphi \\ dy^2 &= \sin^2\theta \cos^2\varphi \, dr^2 + r^2 \cos^2\theta \cos^2\varphi \, d\theta^2 + r^2 \sin^2\theta \sin^2\varphi \, d\varphi^2 \\ &\quad + 2r \sin\theta \cos\theta \cos^2\varphi \, dr d\theta - 2r \sin^2\theta \cos\varphi \sin\varphi \, dr d\varphi - 2r^2 \cos\theta \sin\theta \sin\varphi \cos\varphi \, d\theta d\varphi \end{aligned}$$

- Use eq. (1) for $a = 3$:

$$\begin{aligned} dz &= \frac{\partial z}{\partial t'} dt' + \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi \\ &= \cos\theta \, dr - r \sin\theta \, d\theta \\ dz^2 &= \cos^2\theta \, dr^2 + r^2 \sin^2\theta \, d\theta^2 - 2r \sin\theta \cos\theta \, dr d\theta \end{aligned}$$

Thus,

$$
\begin{aligned}
ds^2 &= -dt'^2 + \underbrace{\left(\sin^2\theta\sin^2\varphi + \sin^2\theta\cos^2\varphi + \cos^2\theta\right)}_{1} dr^2 \\
&\quad + \underbrace{\left(r^2\cos^2\theta\sin^2\varphi + r^2\cos^2\theta\cos^2\varphi + r^2\sin^2\theta\right)}_{r^2} d\theta^2 \\
&\quad + \underbrace{\left(r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi\right)}_{r^2\sin^2\theta} d\varphi^2 \\
&\quad \underbrace{\left(2r\sin\theta\cos\theta\sin^2\varphi + 2r\sin\theta\cos\theta\cos^2\varphi - 2r\sin\theta\cos\theta\right)}_{0} drd\theta \\
&\quad \underbrace{\left(2r\sin^2\theta\sin\varphi\cos\varphi - 2r\sin^2\theta\cos\varphi\sin\varphi\right)}_{0} drd\varphi \\
&\quad \underbrace{\left(2r^2\cos\theta\sin\theta\sin\varphi\cos\varphi - 2r^2\cos\theta\sin\theta\sin\varphi\cos\varphi\right)}_{0} d\theta d\varphi \\
ds^2 &= -dt'^2 + dr^2 + r^2 d\theta^2 + r^2\sin^\theta d\varphi.
\end{aligned}
$$

2. [7 points] Spherical coordinates $x^a = (t, r, \theta, \varphi)$, new coordinates $x'^a = (u, R, \Theta, \phi)$

- Use eq. (1) for $a = 1$, $a = 2$, $a = 3$:

$$dr = dR, \quad d\theta = d\Theta, \quad d\varphi = d\phi$$

- Use eq. (1) for $a = 0$:

$$dt = du + \frac{dr*}{dR} dR$$

with

$$
\begin{aligned}
\frac{dr*}{dR} &= 1 + r_0 \frac{1/r_0}{R/r_0 - 1} \\
&= 1 + \frac{r_0}{R - r_0} \\
&= \frac{R - r_0 + r_0}{R - r_0} \\
&= \frac{R}{R - r_0} \\
&= \frac{1}{1 - r_0/R} = \frac{1}{F(R)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
dt &= du + \frac{1}{F} dR \\
dt^2 &= du^2 + \frac{1}{F^2} dR^2 + \frac{2}{F} dudR.
\end{aligned}
$$

Finally

$$
\begin{aligned}
ds^2 &= -F\left(du^2 + \cancel{\frac{1}{F^2}dR^2} + \frac{2}{F}dudR\right) + \cancel{\frac{1}{F}dR^2} + R^2 d\Theta^2 + R^2 \sin^2\Theta d\phi^2 \\
&= -F du^2 - \frac{2}{F}dudR + R^2 d\Theta^2 + R^2 \sin^2\Theta d\phi^2
\end{aligned}
$$

3. [7 points] Expand general form of line element

$$
\begin{aligned}
ds^2 &= g_{ab}\, dx^a dx^b \\
&= g_{00}\,(dx^0)^2 + g_{01}\, dx^0 dx^1 + g_{02}\, dx^0 dx^2 + g_{03}\, dx^0 dx^3 \\
&= +g_{10}\, dx^1 dx^0 + g_{11}\,(dx^1)^2 + g_{12}\, dx^1 dx^2 + g_{13}\, dx^1 dx^3 \\
&= +g_{20}\, dx^2 dx^0 + g_{21}\, dx^2 dx^1 + g_{22}\,(dx^2)^2 + g_{23}\, dx^2 dx^3 \\
&= +g_{30}\, dx^3 dx^0 + g_{31}\, dx^3 dx^1 + g_{32}\, dx^2 dx^2 + g_{33}\,(dx^3)^2.
\end{aligned}
$$

(a) For $x^a = (u, R, \Theta, \phi)$

$$
\begin{aligned}
ds^2 &= g_{00}\,(du)^2 + g_{01}\, dudR + g_{02}\, dud\Theta + g_{03}\, dud\phi \\
&= +g_{10}\, dRdu + g_{11}\,(dR)^2 + g_{12}\, dRd\Theta + g_{13}\, dRd\phi \\
&= +g_{20}\, d\Theta du + g_{21}\, d\Theta dR + g_{22}\,(d\Theta)^2 + g_{23}\, d\Theta d\phi \\
&= +g_{30}\, d\phi du + g_{31}\, d\phi dR + g_{32}\, d\phi d\Theta + g_{33}\,(d\phi)^2.
\end{aligned}
$$

Compare with eq. (2) in problem set 4. ATTENTION: mixed terms contribute twice: e.g., $-2e^{2\beta}dudR = \underbrace{-e^{2\beta}}_{g_{01}} dudR \underbrace{-e^{2\beta}}_{g_{10}} dRdu$. For simplicity, let $U = \left(\frac{f}{R}e^{2\beta} - g^2 R^2 e^{2\alpha}\right)$

$$
g_{ab} = \begin{pmatrix} -U & -e^{2\beta} & -gR^2e^{2\alpha} & 0 \\ -e^{2\beta} & 0 & 0 & 0 \\ -gR^2e^{2\alpha} & 0 & R^2e^{2\alpha} & 0 \\ 0 & 0 & 0 & R^2e^{-2\alpha}\sin^2\Theta \end{pmatrix}
$$

(b) Solution from exercise 2 reads

$$
g_{ab} = \begin{pmatrix} -F & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\Theta \end{pmatrix}.
$$

Comparing with solution from 3.(a), one gets $\alpha = \beta = g = 0$ and $f(R) = R\,F(R)$. **(c)** Inverse Matrix

$$
g^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & F & 0 & 0 \\ 0 & 0 & R^{-2} & 0 \\ 0 & 0 & 0 & R^{-2}\sin^{-2}\Theta \end{pmatrix}.
$$

4. [3 points] Metric and its inverse

$$
g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}.
$$

(a) Given $X^a = (1, r, r^2)$, calculate $X_a = g_{ab}X^b$. Expressing it as matrix vector multiplication

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$$
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ r \\ r^2 \end{pmatrix}$$
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 \\ r^3 \\ r^4\sin^2\theta \end{pmatrix}$$

(b) Given $Y_a = (0, -r^2, r^2\cos^2\theta)$, calculate $Y^a = g^{ab}Y_b$. Expressing it as matrix vector multiplication

$$\begin{aligned} \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} &= \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \begin{pmatrix} 1 \\ -r^2 \\ r^2\cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ \cot^2\theta \end{pmatrix} \end{aligned}$$

4. [5 points] Metric and its inverse $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$. Given $V^a = (1, 1)$ and $W_a = (0, 2)$. First approach: $V \cdot W = g_{ab}V^aW^b$. One needs $W^b = g^{ba}W_a$. Expressing it as matrix vector multiplication

$$\begin{pmatrix} W^1 \\ W^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2r^{-2} \end{pmatrix}$$

Thus[1].

$$\begin{aligned} V \cdot W &= g_{ab}V^aW^b \\ &= g_{11}V^1\cancelto{0}{W^1} + \cancelto{0}{g_{12}}V^1W^2 + \cancelto{0}{g_{21}}V^2W^1 + g_{22}V^2W^2 \\ &= 1\,r^2\,r^{-2}2 = 2. \end{aligned}$$

Second approach $V \cdot W = V^aW_a = V^1W_1 + V^2W_2 = 2$.
Both methods must agree because

$$g_{ab}V^aW^b = V^a \underbrace{g_{ab}W^b}_{W_a} = V^aW_a$$

[1]Note that the expression $g_{ab}V^aW^b$ can be expressed as

$$g_{ab}V^aW^b = \begin{pmatrix} W^1 & W^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix},$$

whereas V^aW_a as

$$V^aW_a = \begin{pmatrix} W_1 & W_2 \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix},$$