MTH6132, Relativity

Problem Set 3

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Index Notation

1. Which of the following expressions have a meaning according to the index conventions discussed in class?

$$(A^i B_i)C_j = (C_j D^j)B_i,$$
$$(A_k B^k)C_j = (A_m B^m)D_j$$
$$A_i B_j C_k D^k E_f = M_i N_j P_f Q_k,$$
$$A_i B_j = A_j B_i$$
$$A_m = \frac{D_k B_m}{\sqrt{C^k B_k}}$$

2. If lower case Latin indices take values 1 and 2, write down all the components of the following quantities in full

$$G_{ij}, \qquad A^i B_i, \qquad \Gamma^i{}_{jk}, \qquad \Gamma^i{}_{ij}, \qquad R^i{}_{jkl}, \qquad R^i{}_i$$

Coordinate transformation and tensor definition

3. **(a)** If A^a is a contravariant vector and B_a is a covariant vector, then show that $A^a B_a$ is a scalar (hint: show that $A'^a B'_a = A^a B_a$)

(b) Let B^i be an arbitrary contravariant vector and $A_i B^i$ a scalar. Show that A_i is a covariant vector (hint: you can start with the result from (a), i.e.,. $A'^i B'_i = A^i B_i$)

(c) Let $Z^a{}_{bc}{}^d$ be a tensor of type (2,2). Show that $Z^a{}_b = Z^a{}_{bc}{}^c$ is also a tensor. What type is $Z^a{}_b$?

4. Let V^a be a contravariant vector. Show that the quantity $B_b{}^a = \dfrac{\partial V^a}{\partial x^b}$ <u>is not</u> a tensor.

(Anti-)symmetric tensors

5. **(a)** Let S^{ab} and A_{ab} be a symmetric and antisymmetric tensor, respectively. Show that $S^{ab} A_{ab} = 0$. Hint: a and b are dummy variables.

(b) Show that in n dimensions a symmetric tensor S_{ab} has $n(n+1)/2$ independent components, whereas an antisymmetric tensor A_{ab} has $n(n-1)/2$ independent components.

(c) Show that any rank 2 tensor can be expressed as the sum of a symmetric and a antisymmetric parts. Hint: the symmetric and antisymmetric part of tensor T_{ij} are defined, respectively, as

$$T_{(ij)} = \frac{T_{ij} + T_{ji}}{2}, \quad T_{[ij]} = \frac{T_{ij} - T_{ji}}{2}.$$

Contravariant and covariant vectors

6. Consider the Euclidian space $\mathbb{R}^2$ with the usual Cartesian coordinates $x^a = (x, y)$. Let A^a and A_a be a contravariant and a covariant vector, respectively. One can interpret the components of such objects in the following way

$$\text{Contravariant Vector}: \begin{array}{lcl} A^1 & \rightarrow & x\text{-component: projection parallel to the } y\text{-axis} \\ A^2 & \rightarrow & y\text{-component: projection parallel to the } x\text{-axis} \end{array}$$

$$\text{Covariant Vector}: \begin{array}{lcl} A_1 & \rightarrow & x\text{-component: projection perpendicular to the } x\text{-axis} \\ A_2 & \rightarrow & y\text{-component: projection perpendicular to the } y\text{-axis.} \end{array}$$

In a Cartesian coordinate system, both procedures lead to the same results so one usually consider the two types as the same objects.

Let us now introduce a new (non-orthogonal) coordinate system $x'^a = (u, v)$ via

$$x = u + \frac{\sqrt{2}}{2}v, \quad y = \frac{\sqrt{2}}{2}v$$
$$u = x - y, \quad v = \sqrt{2}y$$

(a) Depict both coordinate systems in the same diagram. Hint: In the x-y plane, the $u-$axis is given by $v = 0$ while the $v-$axis by $u = 0$.

(b) In the coordinate system x^a, consider a contravariant and a covariant vector with components $A^a = (2, 1)$ and $A_a = (2, 1)$, respectively. Draw these vectors in the diagram and then visualize the corresponding components $A^{a'}$ and $A_{a'}$ in the coordinates x'^a via

$$\text{Contravariant Vector}: \begin{array}{lcl} A^{1'} & \rightarrow & u\text{-component: projection parallel to the } v\text{-axis} \\ A^{2'} & \rightarrow & v\text{-component: projection parallel to the } u\text{-axis} \end{array}$$

$$\text{Covariant Vector}: \begin{array}{lcl} A_{1'} & \rightarrow & u\text{-component: projection perpendicular to the } u\text{-axis} \\ A_{2'} & \rightarrow & v\text{-component: projection perpendicular to the } v\text{-axis.} \end{array}$$

Can you already predict some of the numerical results?

(c) Calculate the values $A^{a'}$ and $A_{a'}$ according to the definition.