MTH6132, Relativity

Problem Set 3

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Index Notation

1. [1 point] Only the second and the third expressions make sense. The first one has free indices not matching. In the third expression the index k appears as a dummy on one side and free on the other. In the fifth expression k appears repeated three times.

2. [1 point] One has that

$$
\begin{aligned}
&G_{ij}: \quad G_{11}, G_{12}, G_{21}, G_{22},\\
&A^iB_i: \quad A^1B_1 + A^2B_2,\\
&\Gamma^i{}_{jk}: \quad \Gamma^1{}_{11}, \Gamma^1{}_{12}, \Gamma^1{}_{21}, \Gamma^1{}_{22}, \Gamma^2{}_{11}, \Gamma^2{}_{12}, \Gamma^2{}_{21}, \Gamma^2{}_{22},\\
&\Gamma^i{}_{ij}: \quad \Gamma^1{}_{11} + \Gamma^2{}_{21}, \Gamma^1{}_{12} + \Gamma^2{}_{22},\\
&R^i{}_{jkl}: \quad R^1{}_{111}, R^1{}_{112}, R^1{}_{121}, R^1{}_{122}, \quad R^2{}_{111}, R^2{}_{112}, R^2{}_{121}, R^2{}_{122},\\
&\qquad\qquad R^1{}_{211}, R^1{}_{212}, R^1{}_{221}, R^1{}_{222}, \quad R^2{}_{211}, R^2{}_{212}, R^2{}_{221}, R^2{}_{222},\\
&R^i{}_i = R^1{}_1 + R^2{}_2.
\end{aligned}
$$

Coordinate transformation and tensor definition

3. (a)[4 points] Given a coordinate transformation $x'^a = x'^a(x^b)$ and the transformation rules for the contravariant vector A^a and the covariant vector B_a

$$
A'^a = \frac{\partial x'^a}{\partial x^b} A^b, \quad B'_a = \frac{\partial x^b}{\partial x'^a} B_b
$$

then

$$
\begin{aligned}
A'^a B'_a &= \left(\frac{\partial x'^a}{\partial x^b} A^b\right)\left(\frac{\partial x^c}{\partial x'^a} B_c\right) \quad \text{[care not to use the same dummy indices in the sums]}\\
&= \underbrace{\frac{\partial x'^a}{\partial x^b}\frac{\partial x^c}{\partial x'^a}}_{\delta^c_b} A^b B_c \quad \text{[re-arraging terms and identifying Kronecker delta]}\\
&= \delta^c_b A^b B_c \quad \text{[exploiting definition of Kronecker delta]}\\
&= A^b B_b \quad \text{[re-label dummy index } b \to a\text{]}\\
&= A^a B_a
\end{aligned}
$$

(b)[2 points] See lecture notes in QMPlus: Lectures 7-9; section 2.5.4.

(c)[4 points] Given a coordinate transformation $x'^a = x'^a(x^b)$ and the transformation rules for tensor of type (2,2)

$$
Z'^a{}_{bc}{}^d = \frac{\partial x'^a}{\partial x^l}\frac{\partial x^m}{\partial x'^b}\frac{\partial x^n}{\partial x'^c}\frac{\partial x'^d}{\partial x^o} Z^l{}_{mn}{}^o
$$

then

$$\begin{aligned}
Z'^a{}_b &= Z'^a{}_{bc}{}^c \quad \text{[Definition of } Z'^a{}_b \text{ in system } x'^a\text{]} \\
&= \frac{\partial x'^a}{\partial x^l}\frac{\partial x^m}{\partial x'^b}\underbrace{\frac{\partial x^n}{\partial x'^c}\frac{\partial x'^c}{\partial x^o}}_{\delta^n_o} Z^l{}_{mn}{}^o \quad \text{[Transf. rule w/ last two indices contracted]} \\
&= \frac{\partial x'^a}{\partial x^l}\frac{\partial x^m}{\partial x'^b}\delta^n_o Z^l{}_{mn}{}^o \quad \text{[exploiting definition of Kronecker delta]} \\
&= \frac{\partial x'^a}{\partial x^l}\frac{\partial x^m}{\partial x'^b}\underbrace{Z^l{}_{mn}{}^n}_{Z^l{}_m} \quad \text{[identify definition of } Z^l{}_m \text{ in system } x^a\text{]} \\
&= \frac{\partial x'^a}{\partial x^l}\frac{\partial x^m}{\partial x'^b} Z^l{}_m \quad \text{[Obtained the transf. rule for tensor } (1,1)\text{].}
\end{aligned}$$

4.[4 points] Given a coordinate transformation $x'^a = x'^a(x^b)$, the transformation rule for the contravariant vector V^a and the chain rule

$$V'^a = \frac{\partial x'^a}{\partial x^b}V^b, \qquad \frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b}$$

then

$$\begin{aligned}
B'^a_b &= \frac{\partial V'^a}{\partial x'^b} \quad \text{[Definition of } B'^a{}_b \text{ in system } x'^a\text{]} \\
&= \frac{\partial x^d}{\partial x'^b}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^c}V^c\right) \quad \text{[Contravariant and chain rule transformations]} \\
&= \frac{\partial x^d}{\partial x'^b}\underbrace{\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^c}\right)}_{\frac{\partial^2 x'^a}{\partial x^d \partial x^c}} V^c + \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\underbrace{\frac{\partial}{\partial x^d}\left(V^c\right)}_{B_d{}^c} \quad \text{[Leibniz Rule]} \\
&= \underbrace{\frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}B_d{}^c}_{\text{usual transf. rule}} + \underbrace{\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^d \partial x^c}V^c}_{\text{extra term}}.
\end{aligned}$$

Thus, $B_b{}^a = \dfrac{\partial V^a}{\partial x^b}$ does not satisfy the transformation rule of a tensor of type $(1,1)$.

(Anti-)symmetric tensors

5. (a)[3 points] Symmetric and antisymmetric tensor: $S^{ab} = S^{ba}$ and $A_{ab} = -A_{ba}$, then

$$\begin{aligned}
S^{ab}A_{ab} &= -S^{ab}A_{ba} \quad \text{[Definition of antisymmetric tensor]} \\
S^{ab}A_{ab} &= -S^{cd}A_{dc} \quad \text{[re-label dummy indices } a \to c,\ b \to d\text{]} \\
S^{ab}A_{ab} &= -S^{dc}A_{dc} \quad \text{[Definition of symmetric tensor]} \\
S^{ab}A_{ab} &= -S^{ab}A_{ab} \quad \text{[re-label dummy indices } d \to a,\ c \to b\text{]} \\
2S^{ab}A_{ab} &= 0 \ \Rightarrow S^{ab}A_{ab} = 0.
\end{aligned}$$

(b)[2 points] In general, an tensor with 2 indices in n dimensions has n^2 independent components. If symmetric $S_{ab} = S_{ba}$, i.e. the lower diagonal terms in the matrix representation are the same as the upper diagonal

$$\begin{pmatrix}
X & X & X & \cdots & X & X \\
-- & X & X & \cdots & X & X \\
-- & -- & X & \cdots & X & X \\
-- & -- & -- & \cdots & -- & X
\end{pmatrix}
\begin{array}{l}
\to \quad n \text{ independent components} \\
\to \ n-1 \text{ independent components} \\
\to \ n-2 \text{ independent components} \\
\to \quad 1 \text{ independent component}
\end{array}$$

Total of independent components is $1 + 2 + \cdots + n = n(n+1)/2$.

Antisymmetric tensor $A_{ab} = -A_{ba}$, thus diagonal terms vanish ($A_{00} = -A_{00}$, $A_{11} = -A_{11}$, ...). There are n diagonal terms. Thus, total of independent components is $n(n+1)/2 - n = n(n-1)/2$.

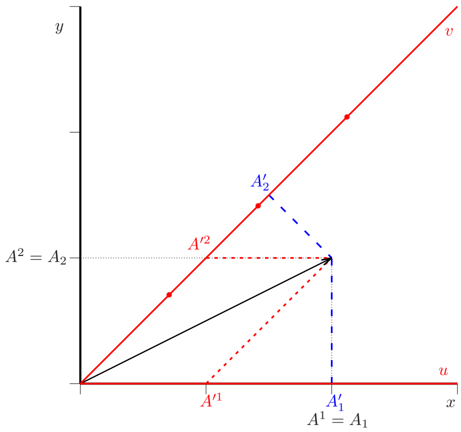
(c) By definition [1 point]

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right), \quad T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right).$$

Adding one finds that $T_{ab} = T_{(ab)} + T_{[ab]}$.

Contravariant and covariant vectors

6. (a)-(b) [4 points]



According to the figure, $A'_1 = A^1 = A_1 = 2$, $A'^1 = 1$; $A'_2 > 2$, $1 < A'^2 < 2$ (rough guess: $A'_2 \sim 2.1$, $A'_2 \sim 1.4$).

(c) [4 points] Coord. Transformation: $x = u + \frac{\sqrt{2}}{2}v$, $y = \frac{\sqrt{2}}{2}v$, $u = x - y$, $v = \sqrt{2}y$

- Contravariant Transformation: $A'^a = \frac{\partial x'^a}{\partial x^b}A^b$, $x'^a = (u, v)$, $x^a = (x, y)$

$$\begin{aligned} A'^1 &= \frac{\partial x'^1}{\partial x^b}A^b \\ &= \frac{\partial x'^1}{\partial x^1}A^1 + \frac{\partial x'^1}{\partial x^2}A^2 \\ &= \frac{\partial u}{\partial x}A^1 + \frac{\partial u}{\partial y}A^2 \\ &= A^1 - A^2 \\ &= 2 - 1 = 1 \end{aligned} \qquad \begin{aligned} A'^2 &= \frac{\partial x'^2}{\partial x^b}A^b \\ &= \frac{\partial x'^2}{\partial x^1}A^1 + \frac{\partial x'^2}{\partial x^2}A^2 \\ &= \frac{\partial v}{\partial x}A^1 + \frac{\partial v}{\partial y}A^2 \\ &= \sqrt{2}A^2 \\ &= \sqrt{2} \approx 1.414 \end{aligned}$$

- Covariant Transformation: $A'_a = \frac{\partial x^b}{\partial x'^a}A_b$, $x'^a = (u, v)$, $x^a = (x, y)$

$$\begin{aligned} A'_1 &= \frac{\partial x^b}{\partial x'^1}A_b \\ &= \frac{\partial x^1}{\partial x'^1}A_1 + \frac{\partial x^2}{\partial x'^1}A_2 \\ &= \frac{\partial x}{\partial u}A_1 + \frac{\partial y}{\partial u}A_2 \\ &= A_1 \\ &= 2 \end{aligned} \qquad \begin{aligned} A'_2 &= \frac{\partial x^b}{\partial x'_2}A_b \\ &= \frac{\partial x^1}{\partial x'^2}A_1 + \frac{\partial x^2}{\partial x'^2}A_2 \\ &= \frac{\partial x}{\partial v}A^1 + \frac{\partial y}{\partial v}A^2 \\ &= \frac{\sqrt{2}}{2}A_1 + \frac{\sqrt{2}}{2}A_2 \\ &= \frac{3\sqrt{2}}{2} \approx 2.121 \end{aligned}$$