MTH6132, Relativity
Problem Set 2
Due 17^{th} October 2018

Hans Bantilan

Parametrized Curves

Consider a point $p \mapsto (x, y, z) = (1, 0, -1)$ and a curve $(x(\lambda), y(\lambda), z(\lambda)) = (\lambda, (1-\lambda)^2, -\lambda)$.

a) Calculate a tangent vector $\vec{u}(\lambda)$ written in frame F at all points along the curve.

b) Using the tangent vector found in part (a), what is $\vec{u}(\lambda)$ written in frame F at point p?

Define a scalar field $f(x, y, z) = x^2 + y^2 - yz$.

c) Calculate $\frac{df}{d\lambda}$ at all points along the curve.

The Twin Paradox Revisited

Let the Sol and Alpha Centauri systems both be at rest in frame $F : (t, x, y, z)$. Let a ship have a worldline $(t(\tau), x(\tau), 0, 0)$ in the frame F. At $\tau = 0$, it is in the Sol system at point $t(0) = 0, x(0) = 1/\alpha$, and it is at rest there in frame F so its four-velocity there is $\vec{u}(0) \mapsto (u^t(0), u^x(0), u^y(0), u^z(0)) = (1, 0, 0, 0)$ written in the frame F. For $\tau > 0$, the ship uniformly accelerates along its worldline with a constant four-acceleration $\vec{a} = (0, \alpha, 0, 0)$ written in its rest frame $\mathcal{F} : (\tau, X, Y, Z)$. By definition, its four-vector written in its rest frame $\mathcal{F}$ is just $\vec{u}(\tau) \mapsto (u^\tau(\tau), u^X(\tau), u^Y(\tau), u^Z(\tau)) = (1, 0, 0, 0)$ along its entire worldline. Notice that this rest frame changes at every point along its worldline due to the constant acceleration i.e. this is a curved worldline.

a) Using $\langle \vec{u}(\tau), \vec{a} \rangle = 0$, $\langle \vec{u}(\tau), \vec{u}(\tau) \rangle = -1$, and $\langle \vec{a}, \vec{a} \rangle = \alpha^2$, show that $a^t = \alpha u^x$ and $a^x = \alpha u^t$.

b) Write the result of (a) as the linear system

$$\begin{aligned} \frac{du^t}{d\tau} &= \alpha u^x \\ \frac{du^x}{d\tau} &= \alpha u^t, \end{aligned}$$

and solve it with initial conditions $u^t(0) = 1, u^x(0) = 0$ to find $u^t(\tau) = \cosh(\alpha\tau)$, $u^x(\tau) = \sinh(\alpha\tau)$.

c) Use the result of (b) with initial conditions $t(0) = 0, x(0) = 1/\alpha$ to show that the worldline of the ship is $t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)$.

The ship travels from the Sol system to the Alpha Centauri system by burning its rear thrusters at a constant acceleration of α. This results in the worldline described by the result in (c), up until the ship reaches the point on its worldline that is halfway between Sol and Alpha Centauri, which let us suppose is at $x(\tau_{halfway}) = \frac{1}{\alpha}\cosh(\alpha)$. After this first leg of the trip described by (c), the ship maneuvers so that its rear is facing its direction of travel, and proceeds to burn its thrusters in exactly the same way it did in the first leg of the trip in order to deccelerate just enough to come to a rest in the F frame at Alpha Centauri. It repeats these two steps on the return journey back to the Sol system: accelerate at constant α, deccelerate at constant α. In all, there are four parts of the journey, the first part of which is described by (c). Let the endpoints of the entire Sol-Alpha Centauri-Sol journey be p and q.

d) What is the elapsed proper time $\Delta\tau$ between p and q experienced by the twin in the ship?

e) What is the elapsed proper time Δt between p and q experienced by the twin at rest in frame F?

f) Given two fixed endpoints p and q, make a conjecture about which worldline maximizes the elapsed proper time experienced by an observer traveling from p to q.