

# Spacetime and Gravity: Assignment 3 Solutions

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In what follows, unless otherwise stated, we will use units such that the speed of light,  $c = 1$ .

1.

We are given the 2 tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (1)$$

and the vector

$$j^\mu = \begin{pmatrix} Q \\ \dot{j}_x \\ \dot{j}_y \\ \dot{j}_z \end{pmatrix} \quad (2)$$

and we are asked to calculate

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (3)$$

So we make use of the summation convention and expand the equation in terms of components of the tensor

$$\partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} = j^\nu \quad (4)$$

Now we pick  $\nu$  manually. Pick  $\nu = 0$  first,

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = j^0 \quad (5)$$

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = Q \quad (6)$$

$$\Rightarrow \nabla \cdot E = Q \quad (7)$$

Now pick  $\nu = 1, 2, 3$  in turn. For  $\nu = 1$ :

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = j^1 \quad (8)$$

$$-\partial_t E_x + \partial_y B_z - \partial_z B_y = j_x \quad (9)$$

$$(10)$$

Similarly,  $\nu = 2$

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = j^2 \quad (11)$$

$$-\partial_t E_y - \partial_x B_z + \partial_z B_x = j_y \quad (12)$$

$$(13)$$

and  $\nu = 3$ ,

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = j^3 \quad (14)$$

$$-\partial_t E_z + \partial_x B_y - \partial_y B_x = j_z \quad (15)$$

$$(16)$$

Adding the last three relations together we obtain

$$-\partial_t \tilde{E} + \nabla \times \tilde{B} = \tilde{j} \quad (17)$$

We have obtained Maxwell's equations! This means that we can recast the whole of Electromagnetism in a special relativistic form, thus making lorentz covariance explicit.

2.

We are given the line element of the unit 2-sphere:

$$ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \quad (18)$$

From which we can extract the metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \quad (19)$$

The inverse metric is then

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\theta)} \end{pmatrix} \quad (20)$$

Using the following relation

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\delta}(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}) \quad (21)$$

We calculate the Christoffel symbols

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\delta}(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}) \quad (22)$$

$$= \frac{1}{2}g^{\alpha 1}(\partial_\gamma g_{1\beta} + \partial_\beta g_{1\gamma} - \partial_1 g_{\beta\gamma}) \quad (23)$$

$$+ \frac{1}{2}g^{\alpha 2}(\partial_\gamma g_{2\beta} + \partial_\beta g_{2\gamma} - \partial_2 g_{\beta\gamma}) \quad (24)$$

$$\Gamma_{\beta\gamma}^1 = \frac{1}{2}g^{11}(\partial_\gamma g_{1\beta} + \partial_\beta g_{1\gamma} - \partial_1 g_{\beta\gamma}) \quad (25)$$

$$\Rightarrow \Gamma_{11}^1 = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \quad (26)$$

$$= \frac{1}{2} \left( \frac{\partial(1)}{\partial\theta} \right) \quad (27)$$

$$= 0 \quad (28)$$

$$\Rightarrow \Gamma_{22}^1 = \frac{1}{2}g^{11}(-\partial_1 g_{22}) \quad (29)$$

$$= -\frac{1}{2} \left( \frac{\partial(\sin^2(\theta))}{\partial\theta} \right) \quad (30)$$

$$= -\sin(\theta) \cos(\theta) \quad (31)$$

$$\Rightarrow \Gamma_{12}^1 = \frac{1}{2}g^{11}(\partial_2 g_{11} + \partial_1 g_{12} - \partial_2 g_{12}) \quad (32)$$

$$= 0 = \Gamma_{21}^1 \quad (33)$$

And

$$\Gamma_{\beta\gamma}^2 = \frac{1}{2}g^{22}(\partial_\gamma g_{2\beta} + \partial_\beta g_{2\gamma} - \partial_2 g_{\beta\gamma}) \quad (34)$$

$$\Rightarrow \Gamma_{22}^2 = \frac{1}{2}g^{22}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) \quad (35)$$

$$= 0 \quad (36)$$

$$\Rightarrow \Gamma_{11}^2 = \frac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) \quad (37)$$

$$= 0 \quad (38)$$

$$\Rightarrow \Gamma_{21}^2 = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{21}) \quad (39)$$

$$= \frac{1}{2 \sin^2(\theta)} \left( \frac{\partial(\sin^2(\theta))}{\partial\theta} \right) \quad (40)$$

$$= \frac{1}{2 \sin^2(\theta)} 2 \sin(\theta) \cos(\theta) \quad (41)$$

$$= \frac{\cos(\theta)}{\sin(\theta)} \quad (42)$$

$$= \cot(\theta) = \Gamma_{12}^2 \quad (43)$$

These are the Christoffel symbols of our space.

## 1 Summary of important concepts

1. We can re-express a classical theory such as Electromagnetism in a form such that its special relativistic nature is manifest. To do this one makes use of the 4-vector notation.

2. Even though in our case we found that the Christoffel symbols are symmetric under exchange of the lower indices:

$$\Gamma_{\beta\gamma}^\alpha = \Gamma_{\gamma\beta}^\alpha \quad (44)$$

this is only true in the case of vanishing torsion.