BSc/MSci Examination

23rd May 2016 14.30

SPA6308 Spacetime and Gravity Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiners:
Prof D. S. Berman
Dr T. Clifton

SECTION A Answer ALL questions in Section A

Question A1

State the principle of equivalence.

[3 marks]

Question A2

Consider a two-dimensional space with the following line-element:

$$ds^2 = f dx^2 + g dy^2 \, .$$

For the vector $\mathbf{V}$, whose components are V^x and V^y, what is the norm, $\mathbf{V}_\mu \mathbf{V}^\mu$?

[5 marks]

Question A3

Show that the line-element

$$ds^2 = -dt^2 + dx^2$$

is invariant under the following transformations

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \qquad \text{and} \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}} \, ,$$

where v is a constant.

[7 marks]

Question A4

Describe how the following three geometric properties change when space becomes curved: the sum of the angles of a triangle; the nature of parallel lines; and the ratio the circumference of a circle to its radius.

[5 marks]

Question A5

Define a geodesic, and use this definition to derive the geodesic equation.

[7 marks]

Question A6

State the cosmological principle.

[3 marks]

Question A7

Write out the Schwarzschild metric and its weak field approximation through a Taylor expansion when $G_N M \ll r$.

[5 marks]

Question A8

What is ds^2 along a light ray? In a space with a line-element given by

$$ds^2 = -(1+x^2)dt^2 + dx^2\,,$$

what is $\frac{dx}{dt}$ along a light-like geodesic?

[5 marks]

Question A9

Write down the Einstein equation and define the symbols you use.

[5 marks]

Question A10

Explain the physical scenario on which Kaluza-Klein theory is based and write down the form of the five dimensional metric.

[5 marks]

SECTION B Answer TWO questions from Section B

Question B1

a) Consider a flat three-dimensional spacetime in polar coordinates (t, r, θ). The line-element is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \,.$$

Write out the metric and find the inverse metric.

[2 marks]

b) The coordinates associated with an observer rotating with constant angular velocity ω are given by (t', r', θ') and are related to the coordinates above as follows:

$$t = t' \,, \qquad r' = r \,, \qquad \theta' = \theta + \omega t \,.$$

Write down the metric and its inverse in these new coordinates.

[6 marks]

c) Using this metric, and the coordinates of the rotating observer, write down explicit expressions for all the Christoffel symbols that have r' as the upper index. You may use:

$$\Gamma^{\sigma}{}_{\alpha\beta} = \frac{1}{2} g^{\sigma\rho} (\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta}) \,.$$

[8 marks]

d) Write the radial component of the geodesic equation.

[5 marks]

e) What, in the “Newtonian limit”, is the force felt by an observer at rest in this rotating coordinate system? What is the name given to this force in non-inertial Newtonian mechanics?

[4 marks]

Question B2

a) Under a change of coordinates $x^\mu \to x'^\mu(x)$ how does the partial derivative ∂_μ transform?

[2 marks]

b) How does a vector field V^μ transform under such a coordinate transformation?

[2 marks]

c) How does $\partial_\mu V^\nu$ transform?

[3 marks]

d) Define a covariant derivative, and then determine how the connection, $\Gamma^\alpha{}_{\mu\nu}$, must transform under such a coordinate transformation.

[8 marks]

e) Write an expression that relates the Riemann curvature tensor to the covariant derivatives acting on a covector field?

[2 marks]

f) Prove the Bianchi identity:

$$D_\rho R^\alpha{}_{\mu\nu\sigma} + D_\nu R^\alpha{}_{\mu\sigma\rho} + D_\sigma R^\alpha{}_{\mu\rho\nu} = 0\,.$$

(Hint: you may use Riemann normal coordinates).

[8 marks]

Question B3

a) Consider a Schwarzschild black hole with Schwarzschild radius, r_s. Using light cones, describe how the causal structure of spacetime changes as a function of the radial distance from the center. For a radially directed light ray, what is $\frac{dr}{dt}$ as a function of the radial distance, r, from the center (in usual Schwarzschild coordinates)?

[6 marks]

b) The temperature of the black hole due to Hawking radiation is given by

$$T = \frac{hc^3}{16\pi^2 GMk} \, .$$

What is the specific heat of this black hole, and why is it unusual?

[2 marks]

c) The Stefan-Boltzmann equation describes the power per unit area, P, emitted by a black-body at temperature T:

$$P = \frac{2\pi^5 k^4}{15c^2h^3} T^4 \, .$$

Work out the lifetime of a black hole that decays through Hawking radiation.

[8 marks]

d) Let us define the “apparent volume” of the black hole as $V(r_s) = 4/3\pi r_s^3$. Now imagine a universe in which the black hole entropy, S, is proportional to the apparent volume as follows:

$$S = kV(r_s)\left(\frac{2\pi c^3}{Gh}\right)^{3/2} \, ,$$

then, assuming the first law of thermodynamics, what would the black hole's temperature have to be as a function of its mass?

[9 marks]

Question B4

The flat FRW universe is described by the following line-element:

$$ds^2 = -dt^2 + R(t)^2(d\sigma^2 + \sigma^2 d\Omega^2_{(2)})\,,$$

where $d\Omega^2_{(2)} = d\theta^2 + \sin^2(\theta)d\phi^2$.

a) For a dust-dominated universe, use the conservation of energy to derive an algebraic equation relating $R(t)$ and the energy density, $\rho(t)$.

[5 marks]

b) The FRW equations governing a flat expanding universe with vanishing cosmological constant, are given by

$$\frac{3\dot{R}^2}{R^2} = 8\pi\rho\,, \qquad \frac{(2\ddot{R}R + \dot{R}^2)}{R^2} = -8\pi p$$

where p is the pressure and dots denote time derivatives. What is the equation of state of a radiation dominated universe? Derive the form of $R(t)$ for such a universe using the equations above.

[8 marks]

c) What is Hubble's law? Derive the relation between Hubble's constant, H, and the scale factor, $R(t)$.

[5 marks]

d) Show that the spatial part of the FRW metric with positive spatial curvature,

$$ds^2_{spatial} = R(t)^2\left(\frac{d\sigma^2}{1-\sigma^2} + \sigma^2 d\Omega^2_{(2)}\right)\,,$$

describes a three-sphere with radius $R(t)$.

[7 marks]

End of Paper