PHY308: Space, Time, and Gravity - Week 1 Homework

Problem 1 (20 marks)

a. The Lorentz transformations are given by:

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}} \tag{1}$$

Show that

$$-t'^2 + x'^2 = -t^2 + x^2 \,. \tag{2}$$

[5 marks]

b. Set $v = \tanh(\zeta)$ and substitute into the Lorentz transformations (1) to find the Lorentz transformations in terms of ζ given by:

$$t' = t\cosh(\zeta) - x\sinh(\zeta) \qquad x' = x\cosh(\zeta) - t\sinh(\zeta)\,. \tag{3}$$

[5 marks]

c. Now again show that

$$-t'^2 + x'^2 = -t^2 + x^2 \tag{4}$$

using the Lorentz transformations (3). [5 marks]

d. Restore the appropriate factors of c in (1), then show that two successive Lorentz transformations in the x-direction, with velocity respectively v_1 and v_2, are equivalent to a Lorentz transformation with velocity $v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$. (Hint: use the matrix form of the Lorentz transformations). What does it happen if you set $v_1 = c$? Explain the phisical meaning of this result. [5 marks]

Problem 2 (6 marks)

Energy, E, and momentum, p, also transform under Lorentz transformations as follows:

$$E' = \frac{E - vp}{\sqrt{1 - v^2}} \qquad p' = \frac{p - vE}{\sqrt{1 - v^2}}. \tag{5}$$

Show that $-E^2 + p^2$ is the quantity that is left invariant by these transformations. Define this invariant quantity to be $-m^2$. In the frame where p=0 and putting back in the factors of c what famous equation have you derived?

Problem 3 (12 marks)

a. Extract the metric, $g_{\mu\nu}$ from the following line element:

$$ds^2 = -f(y)dt^2 + 2f(y)\gamma dxdt + f(y)dx^2 + dy^2 + dz^2, \tag{6}$$

with γ a constant. What is $g^{\mu\nu}$? [6 marks]

b. Show that in the new coordinates (T,X,y,z) given by

$$T = \sqrt{1+\gamma^2}\,t \qquad X = x + \gamma t \tag{7}$$

the metric is diagonal. Hint: calculate $-dT^2 + dX^2$. [6 marks]

Problem 4 (12 marks)

Consider a two dimensional space with coordinates $x^\mu = (t, x)$, and take the line element to be:

$$ds^2 = -f(t,x)dt^2 + 2g(t,x)dtdx + h(t,x)dx^2 \tag{8}$$

a. Write out the metric. Then simply write out the full expressions for

$$x^\mu x_\mu \tag{9}$$

[6 marks]

b. Given the vector $E^\mu = (E, p)$ what is:

$$E_\mu \qquad E_\mu x^\mu \qquad E^\nu E_\nu \qquad x^\mu E_\nu x^\nu E_\mu \qquad x^\mu E^\nu x_\mu E_\nu \tag{10}$$

[6 marks]